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Quasi-interpolation operators based on the trivariate seven-direction C^2 quartic box spline

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Abstract This paper investigates the space $\mathscr{S}(X)$ generated by the integer translates of the trivariate C^2 quartic box spline B defined by a set X of seven directions that forms a regular partition of the space into tetrahedra.

In $\mathscr{S}(X)$ local spline quasi-interpolants are defined by linear combinations of integer translates of B $\{B_\alpha, \alpha \in \mathbb{Z}^3\}$ and local linear functionals $\{\mathbb{D}f(\alpha),\ \alpha \in \mathbb{Z}^3\}$. First a quasi-interpolant of differential type is proposed and its coefficient functionals are linear combinations of values of f with its partial derivatives at the center of the support of B_α. Then, by convenient discretisations of the above differential coefficients, three quasi-interpolants of discrete type with different properties are defined. In this case the coefficient functionals are linear combinations of values of f at specific points in the support of B_α.

Upper bounds both for the norm of the discrete quasi-interpolation operators and for the approximation error are given.

Finally, some numerical examples, illustrating the approximation properties of the proposed quasi-interpolants, are presented.

Keywords Trivariate box spline · Quasi-interpolation operator · Spline approximation

Mathematics Subject Classification (2000) 65D07 · 41A05

1 Introduction

The construction of non-discrete models from given discrete data on volumetric grids is an important problem in many applications, such as scientific visualization and medical imaging (see e.g. [22, 27, 32] and the references therein).

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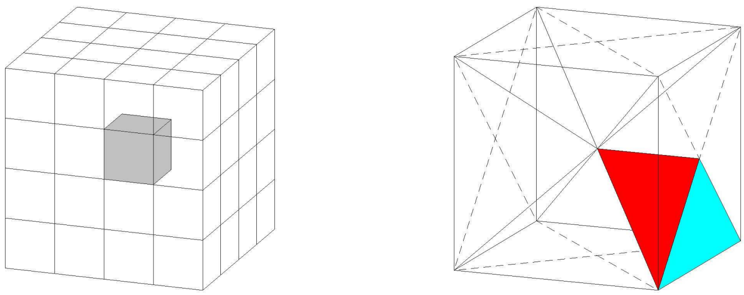
Standard approaches are based on trivariate tensor-product splines (see e.g. [6, 21]), like trilinear continuous splines, where the degree of the polynomial pieces is three, and triquadratic and tricubic splines, where the degree of the polynomial pieces is six and nine, respectively. The first model is not smooth, while triquadratic and tricubic tensor-product splines can be used to construct smooth models of the data. However, these are piecewise polynomials of high degrees and both schemes usually require (approximate) derivative data at certain prescribed points.

Therefore, in the literature, alternative smooth spline models using only data values on the volumetric grid have been proposed.

A local and smooth model of the above kind for the reconstruction of volume data, close to tensor-product schemes, is represented by blending sums of univariate and bivariate C^1 quadratic spline quasi-interpolants (see e.g. [26, 29, 30]).

Another approach is proposed in [22, 27, 32], where approximating schemes based on quadratic or cubic C^1 splines on type-6 tetrahedral partitions are described. The quasi-interpolating piecewise polynomials are directly determined by applying local averaging rules of the data values to compute their Bernstein-Bézier coefficients (BB-coefficients).

The type-6 tetrahedral partitions are uniform partitions of $\mathbb{R}^3$ obtained from a given cube partition (see Fig. 1.1(a)) of the space by subdividing each cube into 24 tetrahedra (see Fig. 1.1(b)). Since cuts with six planes are required to subdivide the cubes in this way, the resulting tetrahedral partition is called type-6 tetrahedral partition [19, 22, 32]. We remark that uniform type partitions provide sometimes the possibility of constructing smooth spline of lower-degree and this is important from a practical point of view for the applications.



(a) (b)

Fig. 1.1 The uniform type-6 tetrahedral partition

Recently, algorithms based on trivariate box splines (see e.g. [7, Chap. 11], [8]) have been proposed as further alternatives to the tensor-product spline schemes for the reconstruction of sampled data in signal processing [12–15, 17]. In particular, in [15], a quasi-interpolation method based on a C^2 quintic box spline is proposed and it achieves the approximation order 4.

In this paper, we construct new quasi-interpolation schemes achieving the same approximation order 4, but they are based on the trivariate C^2 quartic box spline

B, proposed in [12] and defined on a type-6 tetrahedral partition. The box spline B can be considered as a three dimensional extension of the well-known Zwart-Powell quadratic box spline (ZP-element) in 2D (see e.g. [7, Chap. 11], [8, Chap. 1], [9, Chap. 3], [33, Chap. 2]). Furthermore, it has the same smoothness C^2 of the quintic box spline proposed in [15], but it has the lower degree four.

In particular, we consider the space $\mathscr{S}(X)$ generated by the integer translates of B and, in $\mathscr{S}(X)$, we can define local spline quasi-interpolants (abbr. QI) of the form

$$Qf = \sum_{\alpha\in\mathbb{Z}^3} \mathbb{D}f(\alpha)B_\alpha,$$

where $\{B_\alpha, \alpha \in \mathbb{Z}^3\}$ is the family of integer translates of B and $\{\mathbb{D}f(\alpha),\, \alpha \in \mathbb{Z}^3\}$ is a family of local linear functionals.

First we propose a QI of differential type (abbr. DQI), whose coefficient functionals are linear combinations of values of f and its partial derivatives at the center of the support of B_α. This differential QI is exact on the space $\mathscr{D}(X)$ of all polynomials contained in $\mathscr{S}(X)$ and we show that $\mathbb{P}_3(\mathbb{R}^3) \subset \mathscr{D}(X)$, but $\mathbb{P}_4(\mathbb{R}^3) \not\subset \mathscr{D}(X)$, where $\mathbb{P}_l(\mathbb{R}^3)$ denotes the space of trivariate polynomials of total degree at most l.

The main problem arising from differential quasi-interpolants is the computation of derivatives. Therefore, by convenient discretisations of the differential coefficients we define three QIs of discrete type (abbr. dQIs), whose coefficient functionals are linear combinations of values of f at specific points in the support of B_α, with different properties. The first one is constructed so that it is exact on the space $\mathscr{D}(X)$, as the differential one. The second one is exact only on the space $\mathbb{P}_3(\mathbb{R}^3)$ and it minimizes an upper bound for its infinity norm. Finally, the third one is constructed so that it is exact on $\mathbb{P}_3(\mathbb{R}^3)$ and in addition it shows some superconvergence properties at specific points of the domain.

With reference to the three proposed discrete quasi-interpolants, we can notice that the first one has a simpler construction and it is obtained by solving a linear system of three equations and three unknowns, derived from imposing the exactness of the dQI on the space $\mathscr{D}(X)$. In the construction of the second one we impose the exactness only on the space $\mathbb{P}_3(\mathbb{R}^3)$, we have a linear system of two equations and three unknowns and we choose the free parameter minimizing the infinity norm of the coefficient functionals that represents an upper bound for the infinity norm of the quasi-interpolant. In the construction of the third one we have more conditions to impose and the functionals are defined using more data points than in the previous cases. However, numerical results show that, in general, the dQI with superconvergence properties achieves the best performances. Furthermore, it is also interesting to remark that the behaviours of the discrete quasi-interpolant with superconvergence properties and the differential one are almost the same, but in the discrete case we only need to evaluate the function f and not its derivatives.

The paper is organized as follows: in Sect. 2, we recall the trivariate C^2 quartic box spline, we introduce the space spanned by its integer translates and we get the space of all polynomials contained in this spline space. In Sect. 3.1 we define the differential quasi-interpolation operator and, in Sect. 3.2, we construct the three discrete quasi-interpolants. In Sect. 4, we give upper bounds both for the norm of the discrete quasi-interpolants and for the approximation error. Finally, in Sect. 5, we present some numerical examples, illustrating the performances of the proposed quasi-interpolants.

2 On the space of trivariate C^2 quartic splines

In this section we use the notations and results of [7–11] and we study the spline space generated by the integer translates of a trivariate C^2 quartic box spline specified by a set of seven directions.

2.1 The trivariate seven-direction C^2 quartic box spline B

A box spline is specified by a set of direction vectors that determine the shape of the support of the box spline and also its continuity properties.

As noticed in [13], the construction of this trivariate seven-direction C^2 quartic box spline is motivated by the construction of the ZP-element, whose directions are those of the edges and diagonals of the unit square. The box spline proposed in [23] is a box spline whose direction vectors form a cube and its four diagonals, thus $\mathbb{R}^3$ is cut into a symmetric regular arrangement of tetrahedra called type-6 tetrahedral partition (see Fig. 1.1).

Following [23], we consider the set of seven direction vectors of $\mathbb{Z}^3$ and spanning $\mathbb{R}^3$

$$X = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7\}$$

defined by

$$\begin{gathered} e_1 = (1,0,0), \quad e_2 = (0,1,0), \quad e_3 = (0,0,1), \quad e_4 = (1,1,1), \\ e_5 = (-1,1,1), \quad e_6 = (1,-1,1), \quad e_7 = (-1,-1,1). \end{gathered} \tag{2.1}$$

We denote the partial derivatives in the directions of X as follows

$$\partial_{e_1} = \partial_1 = \frac{\partial}{\partial x_1}, \; \partial_{e_2} = \partial_2 = \frac{\partial}{\partial x_2}, \; \partial_{e_3} = \partial_3 = \frac{\partial}{\partial x_3},$$

and

$$\begin{gathered} \partial_{e_4} = \partial_4 = \partial_1 + \partial_2 + \partial_3, \; \partial_{e_5} = \partial_5 = -\partial_1 + \partial_2 + \partial_3, \\ \partial_{e_6} = \partial_6 = \partial_1 - \partial_2 + \partial_3, \; \partial_{e_7} = \partial_7 = -\partial_1 - \partial_2 + \partial_3. \end{gathered}$$

According to [7, Chap. 11] and [8, Chap. 1], since the set X has seven elements and the domain is $\mathbb{R}^3$, the box spline $B(\cdot) = B(\cdot|X)$ is of degree four. The continuity of the resulting box spline depends on the determination of the number d, such that $d+1$ is the minimal number of directions that needs to be removed from X to obtain a reduced set that does not span $\mathbb{R}^3$: then one deduces that the continuity class is C^{d-1}. With the notation given in [7, Chap. 11],

$$d = \min\{|Y| : \; Y \in \mathscr{Y}\} - 1, \tag{2.2}$$

where

$$\mathscr{Y} = \mathscr{Y}(X) = \{Y \subset X : \; \langle X \backslash Y \rangle \neq \mathbb{R}^3\}. \tag{2.3}$$

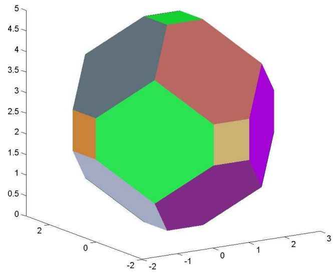


Fig. 2.1 The support of the seven directional box spline

In our case $d=3$, thus the polynomial pieces defined over each tetrahedron are of degree four and they are joined with C^2 smoothness.

The support of the C^2 quartic box spline B is the truncated rhombic dodecahedron centered at the point $(\frac{1}{2},\frac{1}{2},\frac{5}{2})$ and contained in the cube $[-2,3]\times[-2,3]\times[0,5]$, see Fig. 2.1. Its projections on the coordinate planes are the octagonal supports of the bivariate C^2 quartic box spline with the following set of directions of $\mathbb{R}^2$: $\{(1,0);(0,1);(1,1);(1,1);(-1,1);(-1,1)\}$, [28,34].

Now we consider the space $\mathscr{S}(X)$ spanned by the integer translates of the box spline B

$$\mathscr{S}(X)=\left\{s=\sum_{\alpha\in\mathbb{Z}^3}c_\alpha B(\cdot-\alpha),\ \ c_\alpha\in\mathbb{R}\right\}.$$

This space is in general a subspace of the whole space $\mathscr{S}_4^2(\mathbb{R}^3)$ of all C^2 quartic splines.

We introduce the scaled spline space $\mathscr{S}_h(X)$ associated with $\mathscr{S}(X)$ ([8, Chap. 3])

$$\mathscr{S}_h(X)=\sigma_h(\mathscr{S}(X))=\{\sigma_h s:\ s\in\mathscr{S}(X)\},$$

which is defined by means of the scaling operator σ_h, $h>0$

$$\sigma_h f: x\mapsto f\left(\frac{x}{h}\right). \tag{2.4}$$

Thus, $\mathscr{S}_h(X)$ is the spline space defined on the refined lattice $h\mathbb{Z}^3$.

We also recall [7, Chap. 3] that the approximation power of $\mathscr{S}(X)$ is the largest r for which

$$\operatorname{dist}(f,\mathscr{S}_h)=O(h^r)$$

for all sufficiently smooth f, with the distance measured in the $L_p(\Omega)$-norm ($1\le p\le\infty$) on some bounded domain Ω with piecewise smooth boundary. From results given in [7, Chap. 3], we know that the approximation power of $\mathscr{S}(X)$ does not exceed $d+1$, with d defined by (2.2).

2.2 The polynomials contained in $\mathscr{S}(X)$

Now we consider the space $\mathscr{D}(X)$ of all polynomials in $\mathscr{S}(X)$.

Theorem 2.1 *The space of polynomials contained in* $\mathscr{S}(X)$ *is* $\mathscr{D}(X) = \mathbb{P}_3(\mathbb{R}^3)\oplus$ $\mathrm{span}\{p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8, p_9\}$, *with*

$$\begin{array}{lll} p_1 = x_1^4, & p_2 = x_1^3x_2 + 3x_1x_2x_3^2, & p_3 = x_1x_2^3 + 3x_1x_2x_3^2, \\ p_4 = x_2^4, & p_5 = x_1^3x_3 + 3x_1x_2^2x_3, & p_6 = x_2^3x_3 + 3x_1^2x_2x_3, \\ p_7 = x_1x_3^3 + 3x_1x_2^2x_3, & p_8 = x_2x_3^3 + 3x_1^2x_2x_3, & p_9 = x_3^4. \end{array} \tag{2.5}$$

Proof From [7, Chap. 11], we know that

- $\mathbb{P}_3(\mathbb{R}^3) \subset \mathscr{D}(X), \qquad \mathbb{P}_4(\mathbb{R}^3) \not\subset \mathscr{D}(X)$,
- $\dim \mathscr{D}(X) = |\mathbb{B}(X)|$,
  where
  $$\mathbb{B}(X) = \{V \subset X : \ |V| = \ \dim\langle V\rangle = 3\}.$$

Since we have

$$\mathbb{B}(X) = \big\{\{e_i, e_j, e_k\},\ i \neq j,\ i,j = 1,\ldots 6\big\} \setminus \\ \{\{e_1,e_6,e_7\},\ \{e_2,e_5,e_7\}, \{e_3,e_5,e_6\}, \{e_1,e_4,e_5\}, \{e_2,e_4,e_6\}, \{e_3,e_4,e_7\}\}.$$

then

$$\dim \mathscr{D}(X) = |\mathbb{B}(X)| = 29.$$

Thus $\mathscr{D}(X)$ is spanned by the twenty generators of $\mathbb{P}_3(\mathbb{R}^3)$ and by nine homogeneous quartic polynomials $p_i \in \mathbb{P}_4(\mathbb{R}^3)$, $i = 1,\ldots,9$.

From the results given in [7, Chap. 11] we have

$$\mathscr{D}(X) = \{f : \ D_Y f = 0 \text{ for all } Y \in \mathscr{Y}(X)\}.$$

In the set $\mathscr{Y}(X)$, defined by (2.3), the subsets Y with the smallest number of elements are

$$\begin{array}{l} Y_1 = \{e_2,e_3,e_4,e_5\},\ Y_2 = \{e_1,e_3,e_4,e_6\},\ Y_3 = \{e_1,e_2,e_4,e_7\}, \\ Y_4 = \{e_2,e_3,e_6,e_7\},\ Y_5 = \{e_1,e_3,e_5,e_7\},\ Y_6 = \{e_1,e_2,e_5,e_6\}, \end{array}$$

consequently we define the six differential operators

$$\begin{array}{l} D_{Y_1} = \partial_2\partial_3((\partial_2+\partial_3)^2 - \partial_1^2),\ D_{Y_2} = \partial_1\partial_3((\partial_1+\partial_3)^2 - \partial_2^2), \\ D_{Y_3} = \partial_1\partial_2(\partial_3^2 - (\partial_1+\partial_2)^2),\ D_{Y_4} = \partial_2\partial_3((\partial_3-\partial_2)^2 - \partial_1^2), \\ D_{Y_5} = \partial_1\partial_3((\partial_3-\partial_1)^2 - \partial_2^2),\ D_{Y_6} = \partial_1\partial_2(\partial_3^2 - (\partial_1-\partial_2)^2). \end{array}$$

Considering a homogeneous quartic polynomial p of the form

$$\begin{aligned} p = {} & d_1x_1^4 + d_2x_1^3x_2 + d_3x_1^2x_2^2 + d_4x_1x_2^3 + d_5x_2^4 + d_6x_1^3x_3 \\ & + d_7x_1^2x_2x_3 + d_8x_1x_2^2x_3 + d_9x_2^3x_3 + d_{10}x_1^2x_3^2 + d_{11}x_1x_2x_3^2 \\ & + d_{12}x_2^2x_3^2 + d_{13}x_1x_3^3 + d_{14}x_2x_3^3 + d_{15}x_3^4, \end{aligned}$$

and imposing the conditions

$$D_{Y_i}p = 0, \qquad i = 1,\ldots,6,$$

we obtain the following equations

$$d_7 = 3d_9 + 4d_{12} + 3d_{14} = 3d_9 - 4d_{12} + 3d_{14},$$

whence $d_{12} = 0$ and $d_7 = 3(d_9 + d_{14})$. Then successively

$$\begin{aligned} d_8 &= 3d_6 + 4d_{10} + 3d_{13} = 3d_6 - 4d_{10} + 3d_{13} \\ d_{11} &= 3d_2 + 4d_3 + 3d_4 = 3d_2 - 4d_3 + 3d_4, \end{aligned}$$

that give

$$d_{10} = d_3 = 0 \quad \text{and} \quad d_8 = 3(d_6 + d_{13}),\ \ d_{11} = 3(d_2 + d_4).$$

Finally, p can be written in the form

$$\begin{aligned} p = {} & d_1 x_1^4 + d_2(x_1^3 x_2 + 3x_1 x_2 x_3^2) + d_4(x_1 x_2^3 + 3x_1 x_2 x_3^2) + d_5 x_2^4 \\ & + d_6(x_1^3 x_3 + 3x_1 x_2^2 x_3) + d_9(x_2^3 x_3 + 3x^2 x_2 x_3) \\ & + d_{13}(x_1 x_3^3 + 3x_1 x_2^2 x_3) + d_{14}(x_2 x_3^3 + 3x_1^2 x_2 x_3) + d_{15} x_3^4. \end{aligned}$$

and we can take as generators of $\mathscr{D}(X)$ the monomials of $\mathbb{P}_3(\mathbb{R}^3)$ and

$$\begin{aligned} & p_1 = x_1^4, \quad p_2 = x_1^3 x_2 + 3x_1 x_2 x_3^2, \quad p_3 = x_1 x_2^3 + 3x_1 x_2 x_3^2, \\ & p_4 = x_2^4, \quad p_5 = x_1^3 x_3 + 3x_1 x_2^2 x_3, \quad p_6 = x_2^3 x_3 + 3x_1^2 x_2 x_3, \\ & p_7 = x_1 x_3^3 + 3x_1 x_2^2 x_3, \quad p_8 = x_2 x_3^3 + 3x_1^2 x_2 x_3, \quad p_9 = x_3^4. \end{aligned}$$

□

3 Quasi-interpolation operators

In the following two sections we study different kinds of quasi-interpolation operators. A quasi-interpolant

$$Q : \mathscr{F} \to \mathscr{S}(X)$$

is a linear operator defined on some functional space $\mathscr{F}$ by an expression of the form

$$Qf = \sum_{\alpha \in \mathbb{Z}^3} \mathbb{D}f(\alpha) B_\alpha$$

with $B_\alpha(x) = B_{(\alpha_1,\alpha_2,\alpha_3)}(x_1,x_2,x_3) = B(x_1 - \alpha_1 + \frac{1}{2}, x_2 - \alpha_2 + \frac{1}{2}, x_3 - \alpha_3 + \frac{5}{2})$ and $\mathbb{D}f(\alpha)$ linear functionals. We also introduce the scaled quasi-interpolation operator

$$Q_h f(x) = \sigma_h Q \sigma_{1/h} f(x), \tag{3.1}$$

where σ_h is defined by (2.4).

In Sect. 3.1 we consider a differential quasi-interpolation operator whose coefficient functionals $\mathbb{D}f(\alpha)$ are defined by a linear combination of values of f and its derivatives at the center of the support of B_α. Instead, in Sect. 3.2, we construct different types of discrete quasi-interpolation operators whose coefficient functionals $\mathbb{D}f(\alpha)$ can be considered as convenient discretisations of the differential one.

3.1 Differential quasi-interpolants

Considering the general method proposed in [10], we construct a differential quasi-interpolant of the form

$$\widehat{Q}f = \sum_{\alpha\in\mathbb{Z}^3} \widehat{\mathbb{D}} f(\alpha) B_\alpha, \tag{3.2}$$

which is exact on the space of polynomials $\mathscr{D}(X)$ contained in $\mathscr{S}(X)$, i.e. $\widehat{Q}p = p$, $\forall p \in \mathscr{D}(X)$.

This method consists in computing the Taylor expansion of the inverse of the Fourier transform of the box spline $B_{(0,0,0)}$ centered at the origin.

Taking into account the expression of the Fourier transform of a box spline (see [8, Chap. 1] and [9, Chap. 2]) and (2.1), we have

$$\begin{aligned}\hat{B}_{(0,0,0)}(y|X) = \hat{B}_{(0,0,0)}(y_1,y_2,y_3|X) &= \prod_{j=1}^{7} \operatorname{sinc}(e_j \cdot y)\\ &= \operatorname{sinc}(y_1)\operatorname{sinc}(y_2)\operatorname{sinc}(y_3)\operatorname{sinc}(y_1+y_2+y_3)\operatorname{sinc}(-y_1+y_2+y_3)\cdot\\ &\quad \cdot\operatorname{sinc}(y_1-y_2+y_3)\operatorname{sinc}(-y_1-y_2+y_3),\end{aligned}$$

where

$$\operatorname{sinc}(t) = \sin(t/2)/(t/2)$$

is the sinus cardinalis function. We compute the Taylor expansion of $\hat{B}^{-1}_{(0,0,0)}$ of order four and, after some algebra, we obtain

$$\begin{aligned}(\hat{B}_{(0,0,0)}(y_1,y_2,y_3|X))^{-1} = 1 &+ \tfrac{5}{24}(y_1^2+y_2^2+y_3^2) + \tfrac{3}{128}(y_1^4+y_2^4+y_3^4)\\ &+ \tfrac{149}{2880}(y_1^2y_2^2+y_1^2y_3^2+y_2^2y_3^2) + O(\|y\|^6),\end{aligned} \tag{3.3}$$

where $\|y\| = (y\cdot y)^{1/2}$, $y = (y_1,y_2,y_3)$, see e.g. [31, Chap. 5].

Therefore, we define

$$\mathbb{D}^* = \sum_{|\beta|\le 4} a_\beta (-i)^{|\beta|} D^\beta,$$

with $D^\beta = \partial_1^{\beta_1}\partial_2^{\beta_2}\partial_3^{\beta_3}$, $\beta_1+\beta_2+\beta_3 = |\beta|$ and the coefficients a_β are given by the Taylor expansion (3.3). Therefore we obtain the differential operator

$$\mathbb{D}^* = I - \frac{5}{24}\Delta + \frac{3}{128}(\partial_1^4+\partial_2^4+\partial_3^4) + \frac{149}{2880}(\partial_1^2\partial_2^2+\partial_1^2\partial_3^2+\partial_2^2\partial_3^2)$$

that can also be expressed as

$$\mathbb{D}^* = I - \frac{5}{24}\Delta + \frac{3}{128}\Delta^2 + \frac{7}{1440}(\partial_1^2\partial_2^2+\partial_1^2\partial_3^2+\partial_2^2\partial_3^2).$$

In fact, the last term $(\partial_1^2\partial_2^2+\partial_1^2\partial_3^2+\partial_2^2\partial_3^2)$ is not necessary, because it is zero on the polynomial space $\mathscr{D}(X)$ contained in $\mathscr{S}(X)$. First, this is obvious on $\mathbb{P}_3(\mathbb{R}^3)$. Second, it is easy to check that $(\partial_1^2\partial_2^2+\partial_1^2\partial_3^2+\partial_2^2\partial_3^2)p_k = 0$ for all p_k, $1\le k\le 9$ defined by (2.5) .

Therefore we define the differential quasi-interpolant $\widehat{Q}$ of type (3.2) with

$$\widehat{\mathbb{D}} f(\alpha) = \left(I - \frac{5}{24}\Delta + \frac{3}{128}\Delta^2 \right) f(\alpha).$$

Then we deduce the representation of the monomials of $\mathbb{P}_3(\mathbb{R}^3)$ and the polynomials p_k, $1 \le k \le 9$, in terms of translates of box splines. For $\gamma = (\gamma_1, \gamma_2, \gamma_3) \in \mathbb{N}^3$, the monomials are denoted by

$$m_\gamma(x_1, x_2, x_3) = x_1^{\gamma_1} x_2^{\gamma_2} x_3^{\gamma_3}$$

and we obtain

$$m_\gamma = \sum_{\alpha \in \mathbb{Z}^3} \widehat{\mathbb{D}} m_\gamma(\alpha) B_\alpha,$$
$$p_k = \sum_{\alpha \in \mathbb{Z}^3} \widehat{\mathbb{D}} p_k(\alpha) B_\alpha, \ \ 1 \le k \le 9,$$

with $\widehat{\mathbb{D}} m_\gamma$ and $\widehat{\mathbb{D}} p_k$ given in Table 3.1, where $|\gamma| = \gamma_1 + \gamma_2 + \gamma_3$.

Table 3.1 Differential coefficients for the expansion of monomials of $\mathbb{P}_3(\mathbb{R}^3)$ and p_k, $1 \le k \le 9$, in terms of translates of box splines

$\lvert\gamma\rvert = 0$	$\widehat{\mathbb{D}} m_{0,0,0} = m_{0,0,0}$	
$\lvert\gamma\rvert = 1$	$\widehat{\mathbb{D}} m_{1,0,0} = m_{1,0,0}$, $\widehat{\mathbb{D}} m_{0,0,1} = m_{0,0,1}$	$\widehat{\mathbb{D}} m_{0,1,0} = m_{0,1,0}$
$\lvert\gamma\rvert = 2$	$\widehat{\mathbb{D}} m_{2,0,0} = m_{2,0,0} - \frac{5}{12}$, $\widehat{\mathbb{D}} m_{0,0,2} = m_{0,0,2} - \frac{5}{12}$, $\widehat{\mathbb{D}} m_{1,0,1} = m_{1,0,1}$,	$\widehat{\mathbb{D}} m_{0,2,0} = m_{0,2,0} - \frac{5}{12}$ $\widehat{\mathbb{D}} m_{1,1,0} = m_{1,1,0}$ $\widehat{\mathbb{D}} m_{0,1,1} = m_{0,1,1}$
$\lvert\gamma\rvert = 3$	$\widehat{\mathbb{D}} m_{3,0,0} = m_{3,0,0} - \frac{5}{4} m_{1,0,0}$, $\widehat{\mathbb{D}} m_{0,0,3} = m_{0,0,3} - \frac{5}{4} m_{0,0,1}$, $\widehat{\mathbb{D}} m_{1,2,0} = m_{1,2,0} - \frac{5}{12} m_{1,0,0}$, $\widehat{\mathbb{D}} m_{1,0,2} = m_{1,0,2} - \frac{5}{12} m_{1,0,0}$, $\widehat{\mathbb{D}} m_{0,1,2} = m_{0,1,2} - \frac{5}{12} m_{0,1,0}$,	$\widehat{\mathbb{D}} m_{0,3,0} = m_{0,3,0} - \frac{5}{4} m_{0,1,0}$ $\widehat{\mathbb{D}} m_{2,1,0} = m_{2,1,0} - \frac{5}{12} m_{0,1,0}$ $\widehat{\mathbb{D}} m_{2,0,1} = m_{2,0,1} - \frac{5}{12} m_{0,0,1}$ $\widehat{\mathbb{D}} m_{0,2,1} = m_{0,2,1} - \frac{5}{12} m_{0,0,1}$ $\widehat{\mathbb{D}} m_{111} = m_{1,1,1}$
p_k, $1 \le k \le 9$	$\widehat{\mathbb{D}} p_1 = m_{4,0,0} - \frac{5}{2} m_{2,0,0} + \frac{9}{16}$, $\widehat{\mathbb{D}} p_3 = p_3 - \frac{5}{2} m_{1,1,0}$, $\widehat{\mathbb{D}} p_5 = p_5 - \frac{5}{2} m_{1,0,1}$, $\widehat{\mathbb{D}} p_7 = p_7 - \frac{5}{2} m_{1,0,1}$, $\widehat{\mathbb{D}} p_9 = m_{0,0,4} - \frac{5}{2} m_{0,0,2} + \frac{9}{16}$	$\widehat{\mathbb{D}} p_2 = p_2 - \frac{5}{2} m_{1,1,0}$ $\widehat{\mathbb{D}} p_4 = m_{0,4,0} - \frac{5}{2} m_{0,2,0} + \frac{9}{16}$ $\widehat{\mathbb{D}} p_6 = p_6 - \frac{5}{2} m_{0,1,1}$ $\widehat{\mathbb{D}} p_8 = p_8 - \frac{5}{2} m_{0,1,1}$

3.2 Discrete quasi-interpolants

In this section we define three different dQIs of the form

$$\widetilde{Q} f = \sum_{\alpha \in \mathbb{Z}^3} \widetilde{\mathbb{D}} f(\alpha) B_\alpha.$$

The first one is constructed so that it is exact on the space $\mathscr{D}(X)$, as the differential one. The second one is exact only on the space $\mathbb{P}_3(\mathbb{R}^3)$ and it minimizes an upper

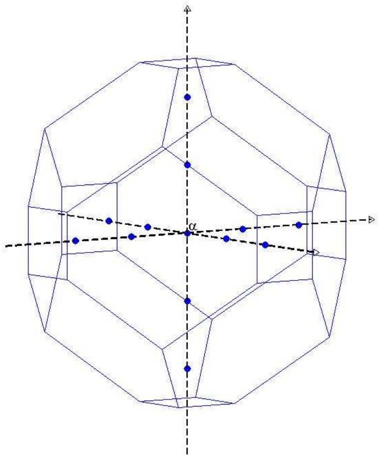


Fig. 3.1 Data points of $\widetilde{\mathbb{D}}^1 f(\alpha)$ and $\widetilde{\mathbb{D}}^2 f(\alpha)$

bound for its infinity norm. The third one is constructed so that it is exact on $\mathbb{P}_3(\mathbb{R}^3)$ and in addition it shows some superconvergence properties at specific points of the domain.

3.2.1 The discrete quasi-interpolant $\widetilde{Q}^1 f$

We consider

$$\widetilde{Q}^1 f = \sum_{\alpha\in\mathbb{Z}^3} \widetilde{\mathbb{D}}^1 f(\alpha) B_\alpha, \tag{3.4}$$

with a discrete coefficient functional $\widetilde{\mathbb{D}}^1 f(\alpha)$ expressed as linear combination of values of f at thirteen points lying in the support of B_α

$$\begin{aligned}\widetilde{\mathbb{D}}^1 f(\alpha) = a_1 f(\alpha) + a_2\left(f(\alpha\pm e_1) + f(\alpha\pm e_2) + f(\alpha\pm e_3)\right)\\ +a_3\left(f(\alpha\pm 2e_1) + f(\alpha\pm 2e_2) + f(\alpha\pm 2e_3)\right),\end{aligned} \tag{3.5}$$

shown in Fig. 3.1.

In order to obtain a discrete quasi-interpolant exact on the space of polynomials $\mathscr{D}(X)$, we require that the discrete coefficient $\widetilde{\mathbb{D}}^1 f(\alpha)$ coincides with the differential one, $\widehat{\mathbb{D}} f(\alpha)$, for $f\in\mathscr{D}(X)$.

For the sake of simplicity we consider the box spline $B_{(0,0,0)}$, i.e. $\alpha = (0,0,0)$ and we impose

$$\widetilde{\mathbb{D}}^1 f(0,0,0) = \widehat{\mathbb{D}} f(0,0,0) \tag{3.6}$$

for all $f\in\mathscr{D}(X)$. From Table 3.1, we deduce

$$\begin{array}{lll}\widehat{\mathbb{D}} m_{0,0,0}(0) = 1, & \widehat{\mathbb{D}} m_{1,0,0}(0) = 0, & \widehat{\mathbb{D}} m_{1,1,0}(0) = 0,\\ \widehat{\mathbb{D}} m_{2,0,0}(0) = -\frac{5}{12}, & \widehat{\mathbb{D}} m_{3,0,0}(0) = 0, & \widehat{\mathbb{D}} m_{2,1,0}(0) = 0,\\ \widehat{\mathbb{D}} m_{1,1,1}(0) = 0, & \widehat{\mathbb{D}} p_1(0) = \frac{9}{16}, & \widehat{\mathbb{D}} p_2(0) = 0,\end{array} \tag{3.7}$$

and similar expressions for the other monomials in $\mathbb{P}_3(\mathbb{R}^3)$ and the other polynomials p_k. Taking into account (3.5), (3.7) and imposing (3.6), there only remain the following three equations

$$a_1 + 6a_2 + 6a_3 = 1, \qquad 2a_2 + 8a_3 = -\frac{5}{12}, \qquad 2a_2 + 32a_3 = \frac{9}{16},$$

whose unique solution is

$$a_1 = \frac{191}{64}, \qquad a_2 = -\frac{107}{288}, \qquad a_3 = \frac{47}{1152}.$$

Thus we define

$$\begin{aligned}\widetilde{\mathbb{D}}^1 f(\alpha) = {} & \tfrac{191}{64} f(\alpha) - \tfrac{107}{288} \left(f(\alpha \pm e_1) + f(\alpha \pm e_2) + f(\alpha \pm e_3) \right) \\ & + \tfrac{47}{1152} \left(f(\alpha \pm 2e_1) + f(\alpha \pm 2e_2) + f(\alpha \pm 2e_3) \right).\end{aligned}$$

3.2.2 The discrete quasi-interpolant $\widetilde{Q}^2 f$

Now we consider a second discrete quasi-interpolant defined by

$$\widetilde{Q}^2 f = \sum_{\alpha \in \mathbb{Z}^3} \widetilde{\mathbb{D}}^2 f(\alpha) B_\alpha, \tag{3.8}$$

and the coefficient functionals $\widetilde{\mathbb{D}}^2 f(\alpha)$ are of the form (3.5), with data points shown in Fig. 3.1. We want a coefficient functional defined using the same number of points of the previous case and we also want to minimize its infinity norm. Therefore we relax the requirements concerning the exactness of the operator. If we require the exactness only on the space $\mathbb{P}_3(\mathbb{R}^3)$, from (3.7), we obtain the two equations

$$a_1 + 6a_2 + 6a_3 = 1, \qquad 2a_2 + 8a_3 = -\frac{5}{12}, \tag{3.9}$$

with three unknowns. In order to obtain the coefficient functional $\widetilde{\mathbb{D}}^2 f(\alpha)$, we choose the free parameter by minimizing its infinity norm that is an upper bound for the infinity norm of the operator $\widetilde{Q}^2$.

From (3.5), it is clear that, for $\|f\|_\infty \leq 1$, $|\widetilde{\mathbb{D}}^2 f(\alpha)| \leq |a_1| + 6|a_2| + 6|a_3|$ and we deduce immediately

$$|\widetilde{Q}^2 f| \leq \sum_{\alpha \in \mathbb{Z}^3} |\, \widetilde{\mathbb{D}}^2 f(\alpha)| B_\alpha \leq \max_{\alpha \in \mathbb{Z}^3} |\widetilde{\mathbb{D}}^2 f(\alpha)|, \tag{3.10}$$

and we conclude

$$\|\widetilde{Q}^2\|_\infty \leq |a_1| + 6|a_2| + 6|a_3|. \tag{3.11}$$

The method used here is similar to the technique given in [1,3–5,16,24,25].

Then, taking into account (3.9), we choose the free parameter minimizing the upper bound for the infinity norm of the operator given in (3.11). We obtain

$$a_1 = \frac{21}{16}, \qquad a_2 = 0, \qquad a_3 = -\frac{5}{96},$$

and we define

$$\widetilde{\mathbb{D}}^2 f(\alpha) = \tfrac{21}{16} f(\alpha) - \tfrac{5}{96} \left(f(\alpha \pm 2e_1) + f(\alpha \pm 2e_2) + f(\alpha \pm 2e_3) \right).$$

3.2.3 *The discrete quasi-interpolant* $\widetilde{Q}^3 f$

Now we construct the third discrete quasi-interpolant of the form

$$\widetilde{Q}^3 f = \sum_{\alpha\in\mathbb{Z}^3} \widetilde{\mathbb{D}}^3 f(\alpha) B_\alpha. \tag{3.12}$$

which shows some additional superconvergence properties.

We know that the approximation power of $\mathscr{S}(X)$ does not exceed 4 and, since we have constructed quasi-interpolants exact at least on $\mathbb{P}_3(\mathbb{R}^3)$ (i.e. $\widetilde{Q}^1$ and $\widetilde{Q}^2$), then the approximation power of $\mathscr{S}(X)$ is 4, and [7, Chap. 3]

$$\left\| f - \widetilde{Q}_h^\nu f \right\|_{L_p(\Omega)} = O(h^4)$$

is valid, with $\nu = 1,2$, $1 \le p \le \infty$, Ω bounded domain in $\mathbb{R}^3$ with piecewise smooth boundary and $\widetilde{Q}_h^\nu$ are the scaled quasi-interpolation operators defined by (3.1).

If we want superconvergence at a specific point M of $\mathbb{R}^3$, i.e. $f(M) - \widetilde{Q}_h^3 f(M) = O(h^5)$, we have to require that, for $f \in \mathbb{P}_4(\mathbb{R}^3)$, the quasi-interpolant $\widetilde{Q}^3$ interpolates the function f at that point.

The specific points where we require superconvergence are the vertices and the centers of each cube of the partition (see Fig. 1.1(a)), i.e. the points:

- $A_{i,j,k} = (i,j,k)$, $(i,j,k) \in \mathbb{Z}^3$,
- $G_{i,j,k} = (i-\frac{1}{2}, j-\frac{1}{2}, k-\frac{1}{2})$, $(i,j,k) \in \mathbb{Z}^3$.

In this case we have a greater number of conditions to impose than in the previous cases, therefore we consider a discrete coefficient functional $\widetilde{\mathbb{D}}^3 f(\alpha)$ based on more data points. Here we consider the coefficient functional

$$\begin{aligned}\widetilde{\mathbb{D}}^3 f(\alpha) = {} & a_1 f(\alpha) + a_2(f(\alpha\pm e_1) + f(\alpha\pm e_2) + f(\alpha\pm e_3)) \\ & + a_3(f(\alpha\pm 2e_1) + f(\alpha\pm 2e_2) + f(\alpha\pm 2e_3)) \\ & + a_4(f(\alpha\pm(e_1+e_2)) + f(\alpha\pm(e_1-e_2)) + f(\alpha\pm(e_1+e_3)) \\ & + f(\alpha\pm(e_1-e_3)) + f(\alpha\pm(e_2+e_3)) + f(\alpha\pm(e_2-e_3))) \\ & + a_5(f(\alpha\pm e_4) + f(\alpha\pm e_5) + f(\alpha\pm e_6) + f(\alpha\pm e_7)),\end{aligned}$$

defined using the thirty-three data points shown in Fig. 3.2.

First we require that the discrete coefficient coincides with the differential one, $\widehat{\mathbb{D}} f(\alpha)$, for $f \in \mathbb{P}_3(\mathbb{R}^3)$ (obtaining the exactness on $\mathbb{P}_3(\mathbb{R}^3)$), then we impose that $\widetilde{Q}^3 f(M) = f(M)$ for $f \in \mathbb{P}_4(\mathbb{R}^3)\backslash\mathbb{P}_3(\mathbb{R}^3)$, the points M being the centers and the vertices of each cube of the partition.

In this case we obtain

$$a_1 = \frac{16871}{4416}, \qquad a_2 = -\frac{507}{736}, \qquad a_3 = \frac{47}{1152}, \qquad a_4 = \frac{1435}{13248}, \qquad a_5 = -\frac{2}{69},$$

and we define

$$\begin{aligned}\widetilde{\mathbb{D}}^3 f(\alpha) = {} & \tfrac{16871}{4416} f(\alpha) - \tfrac{507}{736}(f(\alpha\pm e_1) + f(\alpha\pm e_2) + f(\alpha\pm e_3)) \\ & + \tfrac{47}{1152}(f(\alpha\pm 2e_1) + f(\alpha\pm 2e_2) + f(\alpha\pm 2e_3)) \\ & + \tfrac{1435}{13248}(f(\alpha\pm(e_1+e_2)) + f(\alpha\pm(e_1-e_2)) + f(\alpha\pm(e_1+e_3)) \\ & + f(\alpha\pm(e_1-e_3)) + f(\alpha\pm(e_2+e_3)) + f(\alpha\pm(e_2-e_3))) \\ & - \tfrac{2}{69}(f(\alpha\pm e_4) + f(\alpha\pm e_5) + f(\alpha\pm e_6) + f(\alpha\pm e_7)).\end{aligned}$$

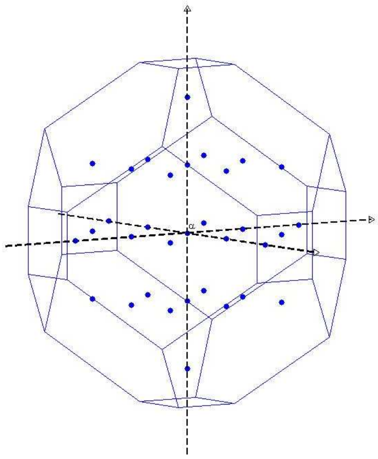


Fig. 3.2 Data points of $\widetilde{\mathbb{D}}^3 f(\alpha)$

4 Norm and error estimates for the discrete quasi-interpolants

In order to study the infinity norms of the proposed discrete quasi-interpolants, we introduce the quasi-Lagrange functions $L_\alpha^{(v)}$, $v = 1,2,3$, by the following expression of $\widetilde{Q}^v$, $v = 1,2,3$,

$$\widetilde{Q}^v f = \sum_{\alpha\in\mathbb{Z}^3} \widetilde{\mathbb{D}}^v f(\alpha) B_\alpha = \sum_{\alpha\in\mathbb{Z}^3} f(\alpha) L_\alpha^{(v)}, \qquad v = 1,2,3,$$

where

$$\begin{aligned}
L_\alpha^{(1)} &= \tfrac{191}{64} B_\alpha - \tfrac{107}{288}(B_{\alpha\pm e_1} + B_{\alpha\pm e_2} + B_{\alpha\pm e_3}) \\
&\quad + \tfrac{47}{1152}(B_{\alpha\pm 2e_1} + B_{\alpha\pm 2e_2} + B_{\alpha\pm 2e_3}), \\
L_\alpha^{(2)} &= \tfrac{21}{16} B_\alpha - \tfrac{5}{96}(B_{\alpha\pm 2e_1} + B_{\alpha\pm 2e_2} + B_{\alpha\pm 2e_3}), \\
L_\alpha^{(3)} &= \tfrac{16871}{4416} B_\alpha - \tfrac{507}{736}(B_{\alpha\pm e_1} + B_{\alpha\pm e_2} + B_{\alpha\pm e_3}) \\
&\quad + \tfrac{47}{1152}(B_{\alpha\pm 2e_1} + B_{\alpha\pm 2e_2} + B_{\alpha\pm 2e_3}) \\
&\quad + \tfrac{1435}{13248}\big(B_{\alpha\pm(e_1+e_2)} + B_{\alpha\pm(e_1-e_2)} + B_{\alpha\pm(e_1+e_3)} + B_{\alpha\pm(e_1-e_3)} \\
&\quad + B_{\alpha\pm(e_2+e_3)} + B_{\alpha\pm(e_2-e_3)}\big) - \tfrac{2}{69}(B_{\alpha\pm e_4} + B_{\alpha\pm e_5} + B_{\alpha\pm e_6} + B_{\alpha\pm e_7}).
\end{aligned} \tag{4.1}$$

We know that the infinity norm of the proposed QIs is equal to the Chebyshev norm of their Lebesgue functions $\|\widetilde{Q}^v\|_\infty = \|\Lambda^v\|_\infty$, $v = 1,2,3$, where $\Lambda^v = \sum_\alpha |L_\alpha^{(v)}|$, and the computation of good upper bounds depends on good upper bounds of $\|\Lambda^v\|_\infty$.

This process is quite complex, however we know that for bounded functions f, a first upper bound for the infinity norm of a discrete quasi-interpolant can be obtained by taking the largest norm of its coefficient functionals. Therefore, we prove

Theorem 4.1 *For the operators $\widetilde{Q}^v$, $v = 1,2,3$, the following bounds hold*

$$\|\widetilde{Q}^1\|_\infty \le \frac{131}{24} \approx 5.46, \quad \|\widetilde{Q}^2\|_\infty \le \frac{13}{8} = 1.625, \quad \|\widetilde{Q}^3\|_\infty \le \frac{5371}{552} \approx 9.73. \tag{4.2}$$

Proof For $\|f\|_\infty \leq 1$, taking into account (3.10) adapted to the three operators $\widetilde{Q}^v$, $v = 1,2,3$, we have that

$$\|\widetilde{Q}^v\|_\infty \leq \|\mathbf{a}_v\|_1, \ \ v = 1,2,3$$

where $\mathbf{a}_v$ is the sequence of parameters associated with $\widetilde{\mathbb{D}}^v f$. Therefore

$$\|\mathbf{a}_1\|_1 = \tfrac{191}{64} + 6\tfrac{107}{288} + 6\tfrac{47}{1152}, \qquad \|\mathbf{a}_2\|_1 = \tfrac{21}{16} + 6\tfrac{5}{96},$$

$$\|\mathbf{a}_3\|_1 = \tfrac{16871}{4416} + 6\tfrac{507}{736} + 6\tfrac{47}{1152} + 12\tfrac{1435}{13248} + 6\tfrac{2}{69},$$

and we obtain (4.2). □

Extending the same logical scheme proposed in [2] to the trivariate case, we obtain a sharper upper bound for the infinity norm of $\widetilde{Q}^v$, $v = 1,2,3$.

Theorem 4.2 *For the operators $\widetilde{Q}^v$, $v = 1,2,3$, the following bounds hold*

$$\|\widetilde{Q}^1\|_\infty \leq \frac{4674}{2323} \approx 2.01, \ \ \|\widetilde{Q}^2\|_\infty \leq \frac{47}{32} \approx 1.47, \ \ \|\widetilde{Q}^3\|_\infty \leq \frac{1167}{493} \approx 2.37. \qquad (4.3)$$

Proof As noticed in [2] for the bivariate case,

$$\|\widetilde{Q}^v\|_\infty = \|\Lambda^v\|_\infty = \max_{(x_1,x_2,x_3)\in T^O} \Lambda^v(x_1,x_2,x_3),$$

where T^O is the tetrahedron with vertices $\left\{\left(\frac{1}{2},\frac{1}{2},\frac{5}{2}\right),(1,0,2),(1,0,3),\left(1,\frac{1}{2},\frac{5}{2}\right)\right\}$. Let $\Omega^v_{T^O}$ be a polyhedral domain including the support of Λ^v and let $\mathscr{T}^v = \{T_r :\ 1 \leq r \leq m^v\}$ be the collection of T^O-like tetrahedra included in $\Omega^v_{T^O}$. Then

$$\begin{aligned}\Lambda^v(x_1,x_2,x_3) &= \sum_{r=1}^{m^v} |L^{(v)}_{\alpha_{(r)}}| \leq \sum_{r=1}^{m^v}\left(\sum_{|\beta|=4}\left|b_{r,\beta}\right|\right) BE^4_\beta(\lambda_1,\lambda_2,\lambda_3,\lambda_4) \\ &= \sum_{|\beta|=4}\left(\sum_{r=1}^{m^v}\left|b_{r,\beta}\right|\right) BE^4_\beta(\lambda_1,\lambda_2,\lambda_3,\lambda_4) \leq \max_{|\beta|=4}\sum_{r=1}^{m^v}\left|b_{r,\beta}\right|,\end{aligned}$$

where $b_{r,\beta} \in \mathbb{R}$ are the BB-coefficients of L^v on T_r, $(\lambda_1,\lambda_2,\lambda_3,\lambda_4)$ are the barycentric coordinates of (x_1,x_2,x_3) with respect to each T_r and $BE^4_\beta(\lambda) = \frac{4!}{\beta!}\lambda^\beta$ ($\lambda = (\lambda_1,\lambda_2,\lambda_3,\lambda_4)$ and $|\beta| = 4$) are the Bernstein polynomials of degree 4 [19, Chap. 15].

We associate with L^v the $m^v \times 35$ matrix $\mathscr{L}^v$ whose rth row contains the BB-coefficients $b_{r,\beta}$, $|\beta| = 4$. Thus, we get as upper bound of $\widetilde{Q}^v$, $v = 1,2,3$

$$\|\mathscr{L}^v\|_1 = \max_{|\beta|=4}\sum_{r=1}^{m^v}\left|b_{r,\beta}\right|.$$

Taking into account (4.1), we compute $\|\mathscr{L}^v\|_1$, for $v = 1,2,3$, and we obtain (4.3). □

Now, we give the following notations:

- let H be a compact set, then, for any function $f \in C(H)$, we denote by $\|f\|_H = \sup\{|f(x,y,z)| : (x,y,z) \in H\}$ the infinity norm of f;
- $D^\beta = D^{\beta_1\beta_2\beta_3} = \partial_1^{\beta_1}\partial_2^{\beta_2}\partial_3^{\beta_3}$, with $\beta_1+\beta_2+\beta_3 = |\beta|$;
- $|f|_{r,B} = \max_{|\beta|=r} \left\|D^\beta f\right\|_B$;
- let $C_{i,j,k}$ be the cube of vertices $A_{i-1,j-1,k-1}$, $A_{i,j-1,k-1}$, $A_{i,j,k-1}$, $A_{i-1,j,k-1}$, $A_{i-1,j-1,k}$, $A_{i,j-1,k}$, $A_{i,j,k}$, $A_{i-1,j,k}$;
- let $\Delta_{i,j,k}$ be the partition of cube $C_{i,j,k}$ into 24 tetrahedra, obtained as explained in Sect. 1;
- let $T \in \Delta_{i,j,k}$ be a tetrahedron, then we set $\Omega_T = \bigcup_{u=i-5}^{i+4}\bigcup_{v=j-5}^{j+4}\bigcup_{w=k-5}^{k+4} C_{u,v,w}$.

Standard results in approximation theory [8,19] allow us to deduce the following theorem.

Theorem 4.3 *Given a tetrahedron T, let $f \in C^4(\Omega_T)$ and $|\gamma| = 0,1,2,3,4$. Then there exist constants $K_{l,|\gamma|} > 0$, $l = 1,2,3$, independent on h, such that*

$$\left\|D^\gamma(f - \widetilde{Q}_h^l f)\right\|_T \le K_{l,|\gamma|} h^{4-|\gamma|} |f|_{4,\Omega_T} .$$

where $\widetilde{Q}_h^l$ are the scaled quasi-interpolation operators defined by (3.1).

A global version of this result follows by taking the maximum over all tetrahedra T.

5 Numerical results

In this section we present some numerical results obtained by a computational procedure developed in a Matlab environment. For the evaluation of box splines we can refer to [18], where an algorithm, that uses the Bernstein-Bézier form of the box spline, is proposed.

We approximate the following functions:

1. the smooth trivariate test function of Franke type

$$\begin{aligned} f_1(x,y,z) = {} & \frac{1}{2}e^{-10((x-\frac{1}{4})^2+(y-\frac{1}{4})^2)} + \frac{3}{4}e^{-16((x-\frac{1}{2})^2+(y-\frac{1}{4})^2+(z-\frac{1}{4})^2)} \\ & + \frac{1}{2}e^{-10((x-\frac{3}{4})^2+(y-\frac{1}{8})^2+(z-\frac{1}{2})^2)} - \frac{1}{4}e^{-20((x-\frac{3}{4})^2+(y-\frac{3}{4})^2)}, \end{aligned}$$

   on the cube $[-\frac{1}{2},\frac{1}{2}]^3$,
2. $f_2(x,y,z) = \frac{1}{9}\tanh(9(z-x-y)+1)$, on the cube $[-\frac{1}{2},\frac{1}{2}]^3$,
3. the Marschner-Lobb function [20]

$$f_3(x,y,z) = \frac{1}{2(1+\beta_1)}\left(1 - \sin\frac{\pi z}{2} + \beta_1\left(1+\cos\left(2\pi\beta_2\cos\left(\frac{\pi\sqrt{x^2+y^2}}{2}\right)\right)\right)\right)$$

   $\beta_1 = \frac{1}{4}$ and $\beta_2 = 6$, on the cube $[-1,1]^3$,
4. $f_4(x,y,z) = \dfrac{\pi y e^{xy}}{40(e-2)}\sin\pi z$, on the cube $[0,1]^3$.

For each test function, defined on the cube $[a,b]^3$, we compute the scaled quasi-interpolants $Q_h f$ ($Q = \widehat{Q}, \widetilde{Q}^1, \widetilde{Q}^2, \widetilde{Q}^3$ defined by (3.2), (3.4), (3.8) and (3.12), respectively) for $h = (b-a)/N$, with $N = 16, 32, 64, 128$. Then, using a $139 \times 139 \times 139$ uniform three-dimensional grid G of points in the domain $[a,b]^3$, we compute the maximum absolute errors $Ef = \max_{(u,v,w)\in G} |f(u,v,w) - Q_h f(u,v,w)|$, for increasing values of N, see Table 5.1. In the table we also report an estimate of the approximation order, rf, obtained by the logarithm to base two of the ratio between two consecutive errors.

We remark that in the construction of functionals associated with each quasi-interpolation operator, we use values of f (and its derivatives in the case of differential quasi-interpolant $\widehat{Q}$) outside the domain.

In [22] the authors propose a quasi-interpolation method for quadratic piecewise polynomials in three variables in BB-form and give some numerical results using the above test functions f_1, f_2, f_3. We denote their quasi-interpolating spline by sq_f and, in the fifth column of Table 5.1, we report the corresponding maximum absolute error and the approximation order estimate.

Furthermore, in [32], the authors propose a local quasi-interpolation method based on cubic C^1 splines on a type-6 tetrahedral partition in three variables in BB-form and give some numerical results with the above test functions f_1 and f_3. We denote their quasi-interpolating spline by sc_f and, in the sixth column of Table 5.1, we report the corresponding maximum absolute error and the approximation order estimate.

Table 5.1 Maximum absolute errors and numerical convergence orders

	$\widehat{Q}f$		$\widetilde{Q}^1 f$		$\widetilde{Q}^2 f$		$\widetilde{Q}^3 f$		sq_f [22]		sc_f [32]	
N	Ef	rf	Ef	rf	Ef	rf	Ef	rf	Ef	rf	Ef	rf
						f_1						
16	4.7(-4)		2.3(-3)		1.1(-2)		7.5(-4)		4.3(-2)		4.3(-2)	
32	1.7(-5)	4.8	1.3(-4)	4.1	8.0(-4)	3.8	1.7(-5)	5.5	1.1(-2)	2.0	1.1(-2)	2.0
64	8.0(-7)	4.4	7.8(-6)	4.1	5.2(-5)	3.9	4.4(-7)	5.3	2.8(-3)	2.0	2.8(-3)	2.0
128	4.6(-8)	4.1	4.7(-7)	4.0	3.3(-6)	4.0	1.9(-8)	4.5	6.9(-4)	2.0	6.9(-4)	2.0
						f_2						
16	2.7(-3)		4.0(-3)		6.2(-3)		2.7(-3)		8.8(-3)			
32	1.3(-4)	4.4	3.9(-4)	3.4	8.2(-4)	2.9	1.4(-4)	4.3	2.4(-3)	1.9		
64	4.9(-6)	4.7	2.7(-5)	3.8	6.9(-5)	3.6	3.9(-6)	5.1	6.3(-4)	2.0		
128	2.1(-7)	4.6	1.7(-6)	4.0	4.7(-6)	3.9	1.1(-7)	5.1	1.6(-4)	2.0		
						f_3						
16	6.2(0)		2.0(-1)		1.9(-1)		2.0(-1)					
32	2.6(-1)	4.6	1.4(-1)	0.5	1.2(-1)	0.7	1.4(-1)	0.5	1.7(-1)		1.7(-1)	
64	6.7(-3)	5.3	1.5(-2)	3.2	4.3(-2)	1.5	1.5(-2)	3.3	1.2(-1)	0.5	1.2(-1)	0.5
128	2.2(-4)	4.9	9.9(-4)	4.0	5.3(-3)	3.0	4.3(-4)	5.1	3.9(-2)	1.6	3.9(-2)	1.6
						f_4						
16	8.7(-7)		6.4(-6)		3.7(-5)		4.4(-7)					
32	5.5(-8)	4.0	4.0(-7)	4.0	2.3(-6)	4.0	2.9(-8)	4.0				
64	3.4(-9)	4.0	2.5(-8)	4.0	1.5(-7)	4.0	1.8(-9)	4.0				
128	2.1(-10)	4.0	1.6(-9)	4.0	9.2(-9)	4.0	1.1(-10)	4.0				

Comparing the discrete quasi-interpolants, we can notice that the overall error is smaller with the operator $\widetilde{Q}^3$, although the bound for its infinity norm (and maybe also the infinity norm itself) is greater than the bound of $\widetilde{Q}^1$ and $\widetilde{Q}^2$. Furthermore using quartic splines the error decreases faster than using the quadratic and cubic C^1 piecewise polynomials proposed in [22,32], respectively, but we have experimentally verified that the computational time is larger.

It is also interesting to remark that the behaviours of the discrete quasi-interpolant $\widetilde{Q}^3$ and the differential one $\widehat{Q}$ are almost the same, but in the discrete case we only need to evaluate the function f and not its derivatives.

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